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THE NON-EUCLIDEAN GEOMETRY.

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IN the third century before Christ there lived the three greatest mathematicians of antiquity — Euclid, Archimedes, and Apollonius. The earliest of the three was Euclid, who was born in the city of Tyre, about 330 B. C., and died about 275 B. C. He was the first professor of mathematics, and at the same time one of the most famous professors in the University of Alexandria, Egypt. There was current in his day an old saying that has appeared almost everywhere and in every age since. Among the French it has taken the form of question and answer. In a discussion of the merits of geometry, one Frenchman puts the question: "Who the devil can learn this?" to which the other replies: "It is the devil who has patience." Euclid's "Elements of Geometry" has been for nearly 2100 years the chief text-book among mathematical teachers. His axioms and propositions have been accepted without question or criticism. But during the last 100 years there has gradually crept into the minds of the greatest thinkers the thought that Euclid is lame in spots. One writer says: "The defects of Euclid as a text-book of geometry have been often stated, and are summed up in de Morgan's article in the "Dictionary of Greek and Roman Biography." The most prominent defects are these: (1) The definitions and axioms contain many assumptions which are not obvious, and in particular the so-called axiom about parallel lines is not self-evident. (2) No explanation is given as to the reason why the proofs take the form in which they are presented; *i. e.*, the synthetical proof is given, but not the analysis by which it was obtained. (3) There is no attempt made to generalize the results arrived at; *e. g.*, the idea of an angle is never extended so as to cover the case where it is equal to or greater than two right angles. (4) The sparing use of superposition as a method of proof. (5) The classification is very imperfect. (6) The work is unnecessarily long and verbose." Hence there has grown up during the last 100 years a demand for axioms and demonstrations that shall be free from objection. The effort has been crowned with success.

It has been conceded in the past that the axioms of geometry could neither be denied nor investigated. Men everywhere found them to agree with their experience, believing that the most rigid reasoning would fail to show any of them untrue. There is no question now that a set of axioms may be taken contradicting in whole or part those of Euclid, and a geometry be built thereon as consistent and as logical as his. Take, for example, one axiom of Euclid, the complicated and unwieldly one whose statement is: "If two lines are cut by a third, and the sum of the interior angles on the same side of the cutting line is less than two right angles, the lines will meet on that side when sufficiently produced." Euclid proved that lines making with a transversal equal alternate angles are parallel; and then, in order to prove that parallels cut by a transversal make alternate angles he used the complicated axiom previously given. Many were the attempts to accomplish this by reasoning about the nature of the straight line and plane angle. Legendre, a celebrated French geometer, attempted it, and in the course of his reasoning proved that the sum of the angles of a triangle can never exceed two right angles, but could not prove that there exists a triangle the sum of whose angles is two right angles.

The time came when some mathematicians began to believe that Euclid's axiom was not capable of proof, and that a geometry could be constructed on the supposition that the axiom is not always true.

As early as 1766 a paper was written by Lambert, in which he maintains that the parallel axiom needs proof, and gives some of the characteristics of geometries in which this axiom does not hold. The greatest mathematician of the nineteenth century, Gauss, sought to prove the axiom of parallels for years, but he never published anything on the subject.

About the year 1830, two men—one a Russian, Lobachevsky, the other a Hungarian, Johann Bolyai—first asserted and then proved that the axiom of parallels is not necessarily true. Thirty years passed before any attention was given to the work of the Russian and the Hungarian. Clifford, an Englishman, one of the most brilliant of mathematicians, in 1870, wrote that "several new ideas have come to me lately: First, I have procured Lobachevsky's 'Geometrical Studies upon the Theory of Parallels,' a small tract, of which Gauss says: 'The author has treated the matter with the hand of a master, and with the true geometrical spirit. I must call your attention to this book, the reading of which will not fail to cause you the most lively pleasure.'"

The "Theory of Parallels" is not a very high-sounding title, but it reveals to us a "new kind of universal space." Gauss, in 1846, says: "The non-Euclidean geometry contains nothing in it that is contradictory, although at first view very many of the results have the air of paradoxes. These apparent contradictions must be regarded as the effect of an illusion, due to the habit we have of considering the Euclidean geometry as rigorous."

From these statements it must not be concluded that because the Euclidean geometry is rigorous that the non-Euclidean is less so. The difficulty with Euclid is in the assumption of axioms or statements that were not capable of proof. Of the non-Euclidean geometries, there are now two well-defined ones: First, that discovered by Lobachevsky, sometimes called the hyperbolic geometry, and second, the elliptic geometry, discovered by Riemann. Riemann studied the foundations of geometry "from a very different point of view, an abstract algebraic point of view, considering not our space and geometrical figures, but a system of variables." He investigated the question, "What is the nature of a function of these variables which can be called element of length or distance? and found that in the simplest cases it must be the square root of a quadratic function of the differentials of the variables whose coefficients may themselves be functions of the variables." To Clifford "we owe the theory of parallels in elliptic space." The lesson to be learned from non-Euclidean geometry is, "that the axioms of geometry are only deductions from our experience, like the theories of physical science." "The assumptions which distinguish the three kinds of geometry, Euclidean, the hyperbolic, and the elliptic, may be expressed in different forms. We may say that one, or two, or no parallels can be drawn through a point; or, that the sum of the angles of a triangle is equal to, less than, or greater than two right angles; or, that a straight line has two real points, one real point, or no real point at infinity; or, that in a plane we can have similar figures, and that a straight line is of finite or infinite length, etc." Any of these forms points out the nature of the geometry.

To illustrate the non-Euclidean geometry of Lobachevsky, let us take his theorem: A straight line maintains its parallelism at all points.

Let AB be parallel to CD at E, and let F be any other point of AB on either side of E, to prove that AB is parallel to CD at F.

Demonstration: To H, on CD, draw EH and FH. If H move off indefinitely on CD, these two lines will approach positions of parallelism with CD. But the limiting position of EH is the line passing through F, and if the limiting position of FH were some other line, FK, F would be the limiting position of H, the intersection of EH and FH.

Another theorem is: If one line is parallel to another, the second is parallel to the first.

Given AB parallel to CD, to prove that CD is parallel to AB.

Demonstration: Draw AC perpendicular to CD. The angle CAB will be acute; therefore, the perpendicular CE from C to AB must fall on that side of A towards which the line AB is parallel to CD (this statement depends on a former proposition not given here). The angle ECD is then acute and less than CEB, which is a right angle. That is, we have CAB less than ACD, and CEB greater than ECD. If the line CE revolve about the point C to the position of CA, the angle at E will decrease to the angle A, and the angle at C will increase to a right angle. There will be some position, say CF, where these two angles become equal; that is CFB = FCD. Draw MN perpendicular to CF at its middle point and revolve the figure about MN as an axis. CD will fall upon the original position of AB, and AB will fall upon the original of CD. Therefore CD is parallel to AB.

Another theorem: In a rectilineal triangle the sum of the three angles cannot be greater than the two right angles.

Demonstration: Suppose in the triangle ABC the sum of the three angles is equal to $180°+a$; then choose in case of the inequality of the sides the smallest BC, halve it in D; draw from A through D the line AD, and make the prolongation of it DE, equal to AD; then join the point E to the point C by the straight line EC. In the congruent triangles ADB and CDE, the angle ABD = DCE,and BAD = DEC; whence follows that in the triangle ACE the sum of the three angles must be equal to $180°+a$; but also the smallest angle BAC of the triangle ABC in passing over into the new triangle ACE has been cut into the two parts EAC and AEC. Continuing this process, continually halving the side opposite the smallest angle we must finally attain to a triangle in which the sum of the three angles is $180° + a$, but wherein are two angles, each of which in absolute magnitude is less than $\frac{1}{2}a$; since now, however, the third angle cannot be greater than 180°, so must a be either null or negative.

Such are some of the methods of demonstration used in the non-Euclidean geometry of Lobachevsky. On the other hand, if we take the non-Euclidean geometry of Riemann, it may be shown that if a straight line is determined by two points, but take the contradictory of the axiom that a straight line is of infinite size; then the straight line returns into itself; but two having intersected get back to that intersection point without ever again meeting. Two intersecting complete straight lines enclose a plane figure, and two such plane figures are congruent if their angles are equal. All complete straight lines are of the same length l. In a given plane all the perpendiculars to a given straight line intersect in a single point, whose distance from the straight line is $\frac{1}{2}l$. Inversely, the locus of all, the points at a distance $\frac{1}{2}l$ on straight lines passing through a given point and lying in a given plane, is a straight line perpendicular to all the radiating lines.

The total volume of the universe, therefore, is equal to l^3/π. The sum of the angles of a plane triangle is greater than a straight angle by an excess proportional to its area.

The greater the area of a triangle, the greater the excess or difference of the angle from 180°.

In conclusion, it may be said that the geometry of Euclid is in no danger of being superseded by either the hyperbolic or the elliptic geometry. The Euclidean geometry, ever since it was given to the world by its author, has been accepted as a geometrical bible, from which the truth, the whole truth and nothing but the truth could be obtained. But as the years rolled on, and the nineteenth century unfolded its years, exceptions began to be taken to Euclid's axioms. The Euclidean axioms are all accepted by the non-Euclidean, excepting the last, which the latter denies and replaces "by its contradictory—that the sum of the angles made on the same side of a transversal by two straight lines may be less than a straight angle without the lines meeting." All of the foregoing is given to illustrate the subject of non-Euclidean geometry, and to call the attention of teachers of geometry to the wonderful method of overcoming the objectionable axiom of Euclid.

Quotations have been made from various writers on the subject, some unchanged and others modified to suit the views of the writer of this paper.